

# Partition Analysis and Symmetrizing Operators

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## Abstract.

Using a symmetrizing operator, we give a new expression for the Omega operator used by MacMahon in Partition Analysis, and given a new life by Andrews and his coworkers. Our result is stated in terms of Schur functions.

In his book "Combinatory Analysis", MacMahon introduced an Omega operator. Recently, Andrews et al [1–3] further developed the theory of Partition Analysis. We show in theorem 4 that the Omega operator can be expressed by a symmetrizing operator. As a consequence, we can formulate:

$$\sum_{\geq} \Omega \lambda^k / \prod_{x \in \mathbb{X}} (1 - x\lambda) \prod_{y \in \mathbb{Y}} (1 - \frac{y}{\lambda})$$

in terms of Schur functions of  $\mathbb{X}$  and  $\mathbb{Y}$  (and therefore in terms of the elementary symmetric functions in  $\mathbb{X}$  and  $\mathbb{Y}$ ).

Recall the definitions of MacMahon's Omega operator  $\Omega$  and of the symmetrizing operator  $\pi_\omega$ .

## Definition 1

$$\sum_{\geq} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1, \dots, s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1, \dots, s_r}.$$

By iteration, it is sufficient to treat the case of one variable  $\lambda$  only .

**Definition 2** [6] Given  $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$  of cardinality  $|\mathbb{X}| = n$ , the symmetrizing operator  $\pi_\omega$  is defined by:

$$\forall f(x_1, \dots, x_n), \pi_\omega f(x_1, \dots, x_n) = \sum_{\sigma \in \mathfrak{S}(\mathbb{X})} \sigma \left( \frac{f(x_1, \dots, x_n)}{\Delta(\mathbb{X})} x_1^{n-1} \cdots x_n^0 \right),$$

writing  $\Delta(\mathbb{X})$  for the Vandermonde  $\prod_{1 \leq i < j \leq n} (x_i - x_j)$ , the sum being over all permutations  $\sigma$  in the symmetric group  $\mathfrak{S}(\mathbb{X})$ .

Recall that complete symmetric functions  $S^j(\mathbb{X})$  are defined by the generating function:

$$\sum_{j=0}^{\infty} S^j(\mathbb{X}) \lambda^j = \frac{1}{\prod_{i=1}^n (1 - x_i \lambda)}.$$

Complete symmetric functions are compatible with union of alphabets (denoted ‘+’). Given  $\mathbb{Y} = \{y_1, y_2, \dots, y_m\}$ , we have:

$$S^n(\mathbb{X} + \mathbb{Y}) = \sum_{k=0}^n S^k(\mathbb{X}) S^{n-k}(\mathbb{Y}).$$

Schur functions have two classical expressions:

$$S_{\mu}(\mathbb{X}) = \left| x_i^{\mu_j + j - 1} \right|_{1 \leq i, j \leq n} / \Delta(\mathbb{X}) = \left| S^{\mu_i - i + j}(\mathbb{X}) \right|_{1 \leq i, j \leq n},$$

where  $\mu = [\mu_1, \dots, \mu_n]$  with  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ .

From the definition of  $\pi_{\omega}$ , we get [6] :

$$\pi_{\omega} x_1^{\mu_1} \dots x_n^{\mu_n} = \left| x_i^{\mu_j + j - 1} \right|_{1 \leq i, j \leq n} / \Delta(\mathbb{X}) = S_{\mu}(\mathbb{X}). \quad (1)$$

This formula is still valid if  $\mu \in \mathbb{Z}^n$ ,  $\mu_1 > -n, \dots, \mu_n > -1$  :

$$\pi_{\omega} x_1^{\mu_1} \dots x_n^{\mu_n} = S_{\mu}(\mathbb{X}), \quad (2)$$

the Schur function  $S_{\mu}$ , still defined as the determinant  $|S^{\mu_i - i + j}|_{1 \leq i, j \leq n}$ , being either null or equal to  $\pm$  a Schur function indexed by a partition.

Symmetrizing first in  $x_2, \dots, x_n$ , one also has, with the same hypotheses on  $\mu$  :

$$\pi_{\omega} x_1^{\mu_1} S_{\mu_2, \dots, \mu_n}(x_2, \dots, x_n) = S_{\mu}(\mathbb{X}). \quad (3)$$

**Lemma 3** *Given  $\mathbb{X}$ ,  $\mathbb{Y}$  and  $k$  such that  $0 \leq k < |\mathbb{X}|$ , then one has:*

$$\pi_{\omega} \left( \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y}) \right) = \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X}) S^j(\mathbb{Y}). \quad (4)$$

*Proof.* Since powers of  $x_1$  range from  $-k$  to  $\infty$ , we can apply (3):

$$\pi_{\omega} \left( \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{Y}) \right) = \sum_{j=0}^{\infty} S_{j-k, 0^{n-1}}(\mathbb{X}) S^j(\mathbb{Y}).$$

The terms such that  $j < k$  are all null, being determinants with two identical rows, and the sum reduces to the expression stated in the lemma. ■

Let us remark that the operator  $\Omega$  relative to  $x_1, \dots, x_n$  can be obtained from the operator  $x_1, \dots, x_{n+r}$ ,  $r \geq 0$  by specializing  $x_{n+1}, \dots, x_{n+r}$  to 0. Therefore we can suppose that  $n$  be bigger than any given integer. This allows us in the following theorem to suppose that  $n > k$ .

**Theorem 4** *Given two alphabets  $\mathbb{X} = \{x_1, x_2, \dots, x_n\}$  and  $\mathbb{Y} = \{y_1, y_2, \dots, y_m\}$  of cardinality  $n$  and  $m$ , let  $\mathbb{B} = 1 + \mathbb{Y} = \{1\} \cup \mathbb{Y}$ . If  $k < n$ , then we have:*

$$\begin{aligned} \pi_\omega \sum x_1^{j-k} S^j(\mathbb{B}) &= \Omega_{\geq} \frac{\lambda^k}{(1-x_1\lambda) \cdots (1-x_n\lambda)(1-\frac{y_1}{\lambda}) \cdots (1-\frac{y_m}{\lambda})} \\ &= \frac{\sum_{\mu} (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-k, \mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})}, \end{aligned} \quad (5)$$

where  $R(1, \mathbb{X}\mathbb{Y})$  is equal to  $\prod_{x \in \mathbb{X}, y \in \mathbb{Y}} (1-xy)$ , and where the sum is over all partitions  $\mu$  (the sum is in fact finite). The vector  $[-k, \mu_1, \dots, \mu_{n-1}]$  is denoted  $-k, \mu$ .

*Proof.* We first recall Cauchy's formula [7, p. 65]:

$$R(1, \mathbb{X}\mathbb{Y}) = \sum_{\mu} (-1)^{|\mu|} S_{\mu}(\mathbb{X}) S_{\mu'}(\mathbb{Y}),$$

where  $\mu \rightarrow \mu'$  is the conjugation of partitions.

$$\begin{aligned} \Omega_{\geq} \sum_{i,j=0}^{\infty} S^i(\mathbb{X}) S^j(\mathbb{Y}) \lambda^{i-j+k} &= \Omega_{\geq} \frac{\lambda^k}{(1-x_1\lambda) \cdots (1-x_n\lambda)(1-\frac{y_1}{\lambda}) \cdots (1-\frac{y_m}{\lambda})} \\ &= \sum_{i=0}^{\infty} S^i(\mathbb{X}) \sum_{j=0}^{i+k} S^j(\mathbb{Y}) = \sum_{i=0}^{\infty} S^i(\mathbb{X}) S^{i+k}(\mathbb{B}) \\ &= \sum_{j=0}^{\infty} S^{j-k}(\mathbb{X}) S^j(\mathbb{B}). \end{aligned}$$

On the other hand, lemma 3 allows us to write this last sum as  $\pi_\omega \left( \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) \right)$ .

We shall now directly compute the action of  $\pi_\omega$  on  $\sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B})$ , denoting  $\mathbb{X} \setminus x_1 = \{x_2, \dots, x_n\}$ .

$$\begin{aligned} \pi_\omega \sum_{j=0}^{\infty} x_1^{j-k} S^j(\mathbb{B}) &= \pi_\omega x_1^{-k} \sum_{j=0}^{\infty} x_1^j S^j(\mathbb{B}) \\ &= \pi_\omega \frac{x_1^{-k}}{R(1, x_1\mathbb{B})} = \pi_\omega \frac{x_1^{-k} R(1, (\mathbb{X} \setminus x_1)\mathbb{B})}{R(1, \mathbb{X}\mathbb{B})} \\ &= \frac{\pi_\omega \left( x_1^{-k} \sum_{\mu} (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{\mu}(\mathbb{X} \setminus x_1) \right)}{R(1, \mathbb{X}\mathbb{B})} \\ &= \frac{\sum_{\mu} (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-k, \mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \end{aligned}$$

and the theorem is proved. ■

The result can be expressed in terms of elementary symmetric functions because  $e_i(\mathbb{B}) = e_i(\mathbb{Y}) + e_{i-1}(\mathbb{Y})$  and Schur functions are determinants in elementary symmetric functions.

In [4, Theorem 2.1], the authors give a “Fundamental Recurrence” for the numerator of (5).

In [5, Theorem 1.4], Guo-Niu Han expresses the Omega operator in terms of Lagrange interpolation:

$$\Omega \frac{\lambda^k}{A(\lambda)B(\lambda)} = \sum_{i=1}^n \frac{x_i^{n-1-k}}{(1-x_i)B(x_i) \prod_{j \neq i} (x_i - x_j)}, \quad (6)$$

where:

$$A(\lambda) = \prod_{i=1}^n (1 - x_i \lambda), B(\lambda) = \prod_{j=1}^m (1 - y_j \lambda).$$

Let us recall the definition [6] of the Lagrange operator  $L_{\mathbb{X}}$ :

**Definition 5**

$$\forall f \in \mathfrak{Sym}(1|n-1), \quad L_{\mathbb{X}} f(x_1, \dots, x_n) = \sum_{x \in \mathbb{X}} \frac{f(x, \mathbb{X} \setminus x)}{R(x, \mathbb{X} \setminus x)},$$

where  $\mathfrak{Sym}(1|n-1)$  is the space of polynomials in  $x_1, x_2, \dots, x_n$ , symmetrical in  $x_2, \dots, x_n$ , and  $R(x, \mathbb{X} \setminus x) = \prod_{x' \in \mathbb{X} \setminus x} (x - x')$ .

We can express the Lagrange operator in terms of  $\pi_{\omega}$ .

**Lemma 6**  $\forall f \in \mathfrak{Sym}(1|n-1)$ , we have:

$$\pi_{\omega} f(x_1, \dots, x_n) = L_{\mathbb{X}} (f(x_1, \dots, x_n) x_1^{n-1}). \quad (7)$$

*Proof.* Elements of  $f(x_1, x_2, \dots, x_n)$  can be written as sums of powers of  $x_1$ , with coefficients symmetrical in  $x_1, \dots, x_n$ . Checking now that

$$L_{\mathbb{X}}(x_1^k x_1^{n-1}) = \pi_{\omega}(x_1^k) = S^k(\mathbb{X}),$$

is immediate. ■

Formula (7) shows that the Lagrange operator in formula (6) can be replaced by  $\pi_{\omega}$ , and therefore [5, Theorem 1.4] is a consequence of theorem 4.

One does not need to suppose that all the  $x_i$ 's be distinct. Indeed, in a Schur function, one may specialize  $x_1, \dots, x_k$  to the same value  $a$ . This is more of a problem in the Lagrange interpolation formula, where one has in that case to use derivatives of different orders.

Let us finish with a small explicit example of the action of  $\pi_\omega$ , for  $n = 2$ ,  $m = 1$ ,  $k = 1$ .

$$\begin{aligned}
\pi_\omega \left( \sum_{j=0}^{\infty} x_1^{j-1} S^j(\mathbb{B}) \right) &= \frac{\sum_{\mu} (-1)^{|\mu|} S_{\mu'}(\mathbb{B}) S_{-1, \mu}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{-S_1(\mathbb{B}) S_{-1, 1}(\mathbb{X}) + S_{1, 1}(\mathbb{B}) S_{-1, 2}(\mathbb{X})}{R(1, \mathbb{X}\mathbb{B})} \\
&= \frac{(1+y) - y(x_1 + x_2)}{(1-x_1)(1-x_2)(1-x_1y)(1-x_2y)} \\
&= \frac{\Omega}{\geq (1-\lambda x_1)(1-\lambda x_2)(1-y/\lambda)}.
\end{aligned}$$

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